Multidisciplinary Design Optimization Under Correlated Uncertainties

Journal:	*Concurrent Engineering: Research and Applications*
Manuscript ID	CERA-16-0051.R1
Manuscript Type:	Standard Article
Date Submitted by the Author:	27-Oct-2016
Complete List of Authors:	Xu, Huanwei; University of Electronic Science and Technology of China, li, wei Xing, Liudong Zhu, Shun-Peng
Keywords:	Collaborative Optimization, Correlated Uncertainties, Ellipsoidal Set, Multidisciplinary Design Optimization, speed reducer
Abstract:	Uncertainty analysis is a hot research topic in Multidisciplinary Design Optimization (MDO) for complex mechanical systems. Existing MDO works typically assume that uncertainties are independent of each other. In real-world engineering systems, however, correlations do exist between different uncertainties. The MDO methods without considering correlations between uncertainties may cause inaccurate and thus misleading optimization results. In this paper, we make contributions by proposing a new MDO approach based on the ellipsoidal set theory to investigate characteristics of correlated uncertainties and incorporate their effects in the MDO through an advanced collaborative optimization method, where the quantitative model of correlated uncertainties is transformed into constrains of subsystems. Both a mathematical example and a case study of an engineering system are provided to illustrate feasibility and validity of the proposed method.

Multidisciplinary Design Optimization Under Correlated Uncertainties

Huanwei Xu[1,*], Wei Li[1], Liudong Xing[2], Shun-Peng Zhu[1]

[1]School of Mechatronics Engineering, University of Electronic Science and Technology of China, Chengdu, Sichuan, 611731

[2]University of Massachusetts, Dartmouth, MA 02747, USA

Abstract: Uncertainty analysis is a hot research topic in Multidisciplinary Design Optimization (MDO) for complex mechanical systems. Existing MDO works typically assume that uncertainties are independent of each other. In real-world engineering systems, however, correlations do exist between different uncertainties. The MDO methods without considering correlations between uncertainties may cause inaccurate and thus misleading optimization results. In this paper, we make contributions by proposing a new MDO approach based on the ellipsoidal set theory to investigate characteristics of correlated uncertainties and incorporate their effects in the MDO through an advanced collaborative optimization method, where the quantitative model of correlated uncertainties is transformed into constrains of subsystems. Both a mathematical example and a case study of an engineering system are provided to illustrate feasibility and validity of the proposed method.

Key words: Collaborative Optimization; Correlated Uncertainties; Ellipsoidal Set; Multidisciplinary Design Optimization; speed reducer

1. Introduction

Modern complex mechanical systems typically contain multiple subsystems with high functional demands and performance requirements. Conventional engineering design methods applied serial design methods, which consider different subsystems in accordance with different design stages. These conventional design methods actually split inherent connections between subsystems, making them difficult in meeting requirements of modern mechanical systems. Multidisciplinary design optimization

* Corresponding author.
E-mail address: hwxu@uestc.edu.cn

(MDO) methods have been developed to overcome this difficulty and are becoming a trend for designing modern complex mechanical systems[1-6].However, the conventional MDO is a deterministic method, which failed to consider uncertainties. In practice, uncertainties can exist due to many factors such as changes in load, material properties, part geometry and operating conditions. Therefore, over the last few years there has been an increased emphasis on the uncertainty analysis in MDO.

Uncertainty-based MDO methods can be roughly categorized into probabilistic and non-probabilistic methods. Among the probabilistic methods, Du and Chen [7] proposed an integrated MDO method encompassing a statistical approach for propagating and mitigating external and internal uncertainties and a multidisciplinary robust design procedure based on system uncertainty analysis (SUA) and concurrent subsystems uncertainty analysis (CSSUA) methods [8].Koch et al. [9] obtained probability characteristics of random variables based on statistical techniques, then estimated system performance characteristics by uncertainty analysis. Among the non-probabilistic methods, Gu et al. [10] investigated uncertainty propagation through a multidisciplinary system and integrated the worst-case estimates of propagated uncertainty into a robust optimization framework. An implicit uncertainty delivery method was further proposed to estimate uncertainty in [11].Harish et al. [12] used the evidence theory to quantify variables of uncertainty for MDO. Chu et al. [13] used the advanced kriging model to control uncertainties from all aspects, provided an effective and interactive process to reduce the uncertainties in conceptual design.

The above-mentioned methods aim at handling different kinds of uncertainties in MDO, but with an assumption that the uncertainties are independent of each other. In many real mechanical design cases, however, uncertainties can be correlated [14-16]. The existing MDO procedures without considering the correlations between uncertainties may lead to inaccurate, thus misleading design results for these systems. In this paper we advance the state-of-the art by proposing a new MDO approach based on the ellipsoidal set theory to handle correlated uncertainties in MDO.

The remainder of this paper is organized as follows: Section 2 presents the concept of correlated uncertainties. In Section 3 a mathematical model based on the ellipsoidal set theory is built to estimate correlated uncertainties. Building upon the constructed mathematical model, Section 4 presents the new MDO method considering correlated uncertainties. In Section 5 the purposed method is illustrated with a mathematical example and a case study of an engineering example system. Section 6 gives conclusions as well as directions for future work.

2.The concept of correlated uncertainties

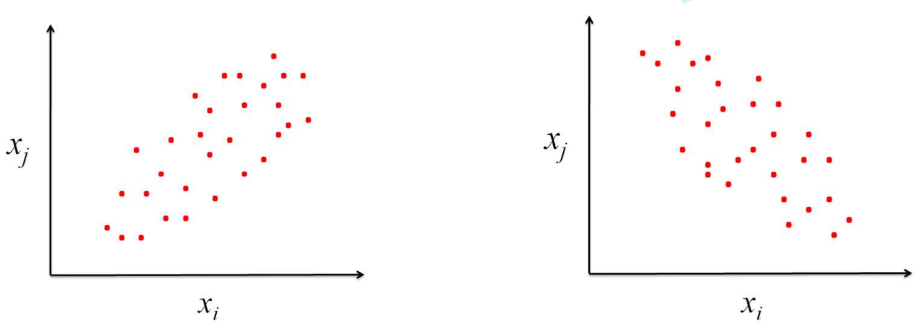
Let x_1, x_2, … ,x_n represent random uncertainties, which can be collectively denoted by a vector $X=(x_1,x_2,\ldots, x_n)^{\mathrm{T}}$. The corresponding covariance matrix C is:

$$C=\begin{pmatrix} D(x_1) & Cov(x_1,x_2) & \cdots & Cov(x_1,x_n) \\ Cov(x_2,x_1) & D(x_2) & \cdots & Cov(x_2,x_n) \\ \vdots & \vdots & \ddots & \vdots \\ Cov(x_n,x_1) & Cov(x_n,x_2) & \cdots & D(x_n) \end{pmatrix} \tag{1}$$

In (1),$D(x_i)$ is the variance of random uncertainty x_i, Cov(x_i,x_j) is the covariance of random uncertainties x_i and x_j. Covariance matrix C depicts correlations between uncertainties. In the case of all random uncertainties are not correlated, C becomes a diagonal matrix. For x_i and x_j, the relation between correlation coefficient $\rho_{x_i x_j}$ and covariance $Cov(x_i,x_j)$ is as follows[17]:

$$\rho_{x_i x_j}=\frac{Cov(x_i,x_j)}{\sigma_{x_i}\sigma_{x_j}}=\frac{E(x_i x_j)-E(x_i)E(x_j)}{\sqrt{D(x_i)}\sqrt{D(x_j)}} \tag{2}$$

where $E(x_i)$ is the mean of x_i, σ_{x_i} is the standard deviation of x_i. The range of correlation coefficient $\rho_{x_i x_j}$ is[-1, 1].In the case of $\rho_{x_i x_j}\in(0,1]$,positive correlation takes place meaning that if x_i increases (decreases),x_j increases (decreases) accordingly (Fig. 1 (a)); in the case of $\rho_{x_i x_j}\in[-1,0)$,negative correlation occurs meaning that if x_i increases (decreases), x_j decreases (increases) accordingly (Fig. 1 (b)).If $\rho_{x_i x_j}$ =0, then x_i and x_j are not correlated, but not necessarily independent.



(a) Positive correlation (b) Negative correlation

Fig. 1 Schematic diagram of correlated uncertainties

3. The model of correlated uncertainties

Ben-Haim and Elishakoff et al. [18] proposed a mathematical model "Bounded-but-Unknown" to evaluate uncertainties in mechanical systems. Ben-Haim [18-20] also studied inherent connections between uncertainties and convexity, and proposed different kinds of convex set models to deal with uncertainties, such as the energy-bound model, envelop-bound model, slope-bound model, and Fourier-bound model. Elements in a convex set are possible functions of uncertain events. In the geometric space, the convex model has a special shape and size. The shape of the convex set represents how much we know about information of uncertainties. The size of the convex set reflects the fluctuation degree of uncertainties [21]. Particularly, the energy-bound model can represent the range of deterioration of quadratic functions; the envelop-bound model is mainly used to describe components dimensions and geometric tolerance; the slope-bound model is often applied to state derivative boundedness of an uncertain function; the Fourier-bound model characterizes spectral coefficients of uncertain phenomena. Ellipsoidal model can be used not only to construct transient energy model and Fourier-bound model, but also to describe static uncertain parameters, which make it have more applications [18]. Therefore, the ellipsoidal model is adapted in this work to describe correlated uncertainties.

In the following subsections, we review basics of the interval model (Section 3.1) and ellipsoidal model (Section 3.2) first, and then propose a quantitative model based on them for modeling correlated uncertainties in Section 3.3.

3.1 Interval model

The upper and lower limits of uncertainty x are x^u,x^l, respectively. The interval model can thus be expressed as:

$$x_p = [x^l, x^u] = \{x \in R \mid x^l \le x \le x^u\} \tag{3}$$

The midpoint of the interval can be calculated by following formula:

$$\bar{x} = mid(x_p) = (x^l + x^u)/2 \tag{4}$$

The radius of the interval is:

$$\Delta x = rad(x_p) = (x^u - x^l)/2 \tag{5}$$

which can reflect the deviation size of uncertain factor x.

An interval model is suitable for circumstances where only upper and lower bounds of uncertainties are known. Since the construction of an interval model requires little information, it is the simplest convex set model and cannot reflect correlations between uncertainties. Hence the uncertainty analysis based on the interval model may lead to wrong results. In practice, the applications of the interval model are restricted.

3.2 Ellipsoidal model

Ellipsoidal model depicts uncertainties with an ellipsoid where the major and minor axises are not parallel with the coordinate axises. Since the ellipsoidal model can approximate most of convex sets, it is more attractive than other convex set models [22].In general, an ellipsoidal model of n dimensional vector X can be expressed as:

$$X \in E = \{X \left| (X - \bar{X})^T Q (X - \bar{X}) \le \varepsilon^2 \right.\} \tag{6}$$

where $\bar{X}$ is the nominal value of X, Q is a characteristic matrix of convex set, which determines the shape and the principal axis direction of the ellipsoid. ε is the radius of the ellipsoid, which describes the size of the ellipsoid as a positive number. Refer to [22-24] for more details of the ellipsoidal model.

The$n$-dimensional uncertainty vector X can be represented by a set D:

$$D = \{X \left| H(x, \varepsilon) \le 0 \right.\}$$

where the m-dimensional vector ($m \le n$) ε is the size of the set, that is, the fluctuation degree of x. When ε=0, the elements in set D only have nominal values, then x is a deterministic parameter, and $H(\bar{x}, 0) = 0$ .When $H(x, \varepsilon)$ is twice differentiable at point $\bar{x}$, the mixed partial derivative is:

$$\left(\frac{\partial^2 H}{\partial x_i \partial \varepsilon_j} \right)_{x=\bar{x},\ \varepsilon=0} = 0; \qquad (i = 1, 2, \cdots, n;\ j = 1, 2, \cdots m) \tag{7}$$

we can obtain:

$$H(x, \varepsilon) \cong \frac{1}{2} (X - \bar{X})^T Q (X - \bar{X}) - J^T \varepsilon \tag{8}$$

where

$$Q=\begin{bmatrix} \frac{\partial^2 H}{\partial x_1 \partial x_1} & \frac{\partial^2 H}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 H}{\partial x_1 \partial x_n} \\ \frac{\partial^2 H}{\partial x_2 \partial x_1} & \frac{\partial^2 H}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 H}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 H}{\partial x_n \partial x_1} & \frac{\partial^2 H}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 H}{\partial x_n \partial x_n} \end{bmatrix}_{(x=\bar{x},\varepsilon=0)} \quad ; J^T=(-\frac{\partial H}{\partial \varepsilon_1},-\frac{\partial H}{\partial \varepsilon_2},\cdots-\frac{\partial H}{\partial \varepsilon_r})_{(x=\bar{x},\varepsilon=0)}$$

When Q is a positive definite matrix, $H(x,\varepsilon)\le 0$ can be approximated as anellipsoidwith a smooth boundary.

3.3 Ellipsoidal model under correlated uncertainties

In this subsection, we propose an integrated model based on the interval model and ellipsoidal model for modeling correlated uncertainties. As defined earlier, $X=(x_1,x_2,\ldots,x_n)^{\mathrm{T}}$ is a vector of random uncertainties. The upper and lower bounds of X are $X^U=(x_1^u,x_2^u,\cdots,x_n^u)^T$ and $X^L=(x_1^l,x_2^l,\cdots,x_n^l)^T$ respectively. The mean of X is:

$$\overline{X}=(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n)^T=(\frac{x_1^u+x_1^l}{2},\frac{x_2^u+x_2^l}{2},\cdots,\frac{x_n^u+x_n^l}{2})^T \tag{9}$$

The ellipsoidal modelunder the condition of correlated uncertainties is as follows:

$$G=(X-\bar{X})^T C^{-1}(X-\bar{X})=\begin{pmatrix} x_1 & - & \bar{x}_1 \\ \vdots & & \vdots \\ x_n & - & \bar{x}_n \end{pmatrix}^T \begin{pmatrix} D(x_1) & Cov(x_1,x_2) & \cdots & Cov(x_1,x_n) \\ Cov(x_2,x_1) & D(x_2) & \cdots & Cov(x_2,x_n) \\ \vdots & \vdots & \ddots & \vdots \\ Cov(x_n,x_1) & Cov(x_n,x_2) & \cdots & D(x_n) \end{pmatrix}^{-1} \begin{pmatrix} x_1 & - & \bar{x}_1 \\ \vdots & & \vdots \\ x_n & - & \bar{x}_n \end{pmatrix} \le \varepsilon^2 \tag{10}$$

When uncertainties (m dimension, $m<n$) are correlated, thedeviations of these uncertainties can be represented by:

$$\delta_i = x_i - \bar{x}_i \quad (i=1,2,\cdots,m) \tag{11}$$

which can be expressed by a vector form:

$$\delta^T = \{\delta_1^T, \delta_2^T, \cdots, \delta_i^T, \cdots, \delta_m^T\} \quad (i = 1, 2, \cdots, m)$$

The deviation range of the $i$thvariable δ_i^T can be represented by an ellipsoid set. Then we can obtain a multi-ellipsoidal model:

$$\delta \in E = \{\delta_i : \delta_i^T C_i^{-1} \delta_i \le \varepsilon_i^2\}, (i = 1, 2, \cdots, m) \tag{12}$$

Where C_i is a covariance matrix for the ith ellipsoidal set, which determines the shape and principal axis direction of ellipsoidal, i.e., describes correlations of uncertainties. ε_i(i=1,2,…, m) is the real constant which is used to limit the size of the ellipsoid. If the value of$\varepsilon_i$is very large, it will lead to the constraint of ellipsoidal model not working in the MDO; if the value of ε_i is very small, it will result in a local optimal solution. So the value of $\varepsilon_i$should be set properly. Refer to [21] for ways to estimate the value of ε_i.

As an illustration, consider an MDO problem with two uncertainties x and y, and $x \in [-1, 1]$, $y \in [1, 5]$.The correlation coefficient between x and y is ρ_{xy}=-0.3. Suppose x and y are both normally distributed, the correlation coefficient matrixρ_0 and covariance matrix C are as follows:

$$\rho_0 = \begin{pmatrix} 1 & -0.3 \\ -0.3 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1/3 & -0.2 \\ -0.2 & 4/3 \end{pmatrix}$$

The formula of ellipsoidal model is:

$$G = (X - \bar{X})^T C^{-1} (X - \bar{X}) = \begin{pmatrix} x-0 \\ y-3 \end{pmatrix}^T \begin{pmatrix} 1/3 & -0.2 \\ -0.2 & 4/3 \end{pmatrix}^{-1} \begin{pmatrix} x-0 \\ y-3 \end{pmatrix} \le \varepsilon^2$$

Calculating corresponding maximum inscribed ellipse (as shown in Fig. 2), we can get the smallest radius of ellipsoid.

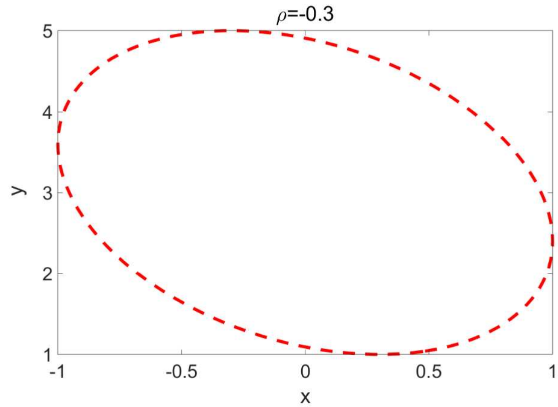


Fig. 2 Maximum inscribed ellipse

4. MDO under correlated uncertainties

In this section, we first review basics of collaborative optimization (CO), and then propose an advanced CO method considering correlated uncertainties.

4.1 Collaborative optimization

CO is an effective MDO method, which decomposes a complex system into several subsystems[25-27].Each subsystem represents a local optimization problem. Correlations or dependencies between subsystems are embodied in the system level. CO can facilitate information organization and management, leading to high computational efficiency.

Suppose the design optimization problem can be decomposed into n subsystems. The CO mathematical expression in the system-level is as follows:

$$\begin{aligned} &\min \quad f(Z) \\ &s.t. \quad J_i(Z)=\sum_{j=1}^{s_i}(z_j-x_{ij}^*)^2=0\,, \quad i=1,2,\ldots,n \end{aligned}$$

where $f(Z)$ is an objective function to evaluate different design solutions concerning factors such as weight, strength, deformation. $J_i(z)$is the ith consistency constraint, Z is the matrix of design variables, z_j is the jth design variable, s_i is the number of design variables for the subject i, x_{ij}^* is the jth optimization result from the ith subsystem, n represents the number of subsystems.

The ith subsystem optimization model is:

$$
\begin{aligned}
&\min \quad J_i(x_i) = \sum_{j=1}^{s_i} (x_{ij} - z_j^*)^2 \\
&s.t. \quad c_i(x_i) \le 0
\end{aligned}
$$

where x_i is a local design variable of the subsystem i, x_{ij} represents the jth variable of subsystem i. z_j^* is jth variable assigned from the system-level to subsystem i, $c_i(x_i)$ is the local constraint.

4.2 CO considering correlated uncertainties

For conventional MDO methods, due to mutual coupling relationships between subsystems, it is difficult to estimate propagating characteristics of correlated uncertainties. But with the help of the quantitative model proposed in Section 3.3, these propagating characteristics can be addressed in the subsystem optimization model. Fig. 4 illustrates the CO framework under the condition of correlated uncertainties, where for the ith subsystem the quantitative model of correlated uncertainties(i.e., Eq. (10)) is transformed into constraint $G_i \le \varepsilon_i^2$.

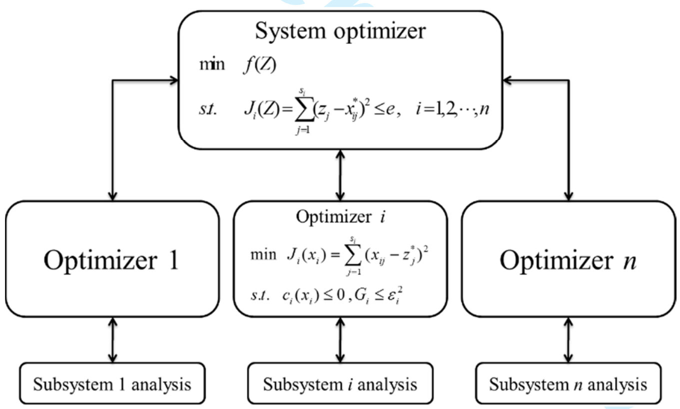


Fig. 3 CO framework of correlated uncertainties

Specifically, the following gives the optimization steps of MDO under the condition of correlated uncertainties.

Step 1. Categorize uncertainties of a system design into two types: correlated uncertainties and uncorrelated uncertainties. For correlated uncertainties, determine the correlation coefficient of any two uncertainties, and then obtain the covariance matrix of correlated uncertainties (Eq. (2)).

Step 2. Establishthe ellipsoidal model using Eq. (10).

Step 3. According to requirements of the real engineering problem and the lower and upper bounds of correlated uncertainties, use the minimum volume method of [23,28] to calculate the minimum radius ε of ellipsoid.

Step 4. Under the CO framework of Fig. 3, decouple a large-scale coupling engineering problem into a series of small-scale, paralleled subsystems considering factors such as different design disciplines, different work teams, or different design stages.

Step5. The ellipsoidal models obtained in step 2 and step 3 are added as constraints to the subsystem optimization models.

Step 6. Utilize the MATLAB fmincon function to solve the optimization problems of the CO framework.

It is worth mentioning that in step 5 the ellipsoidal models should be transformed to constraints. However, Many scholars may be inclined to use finite element method (FEM) to compute stiffness, stress and vibration (performance function) of complex mechanical systems. Since FEM is numerical computation method, it is very difficult to apply our proposed method directly. In such cases, the surrogate model may be suitable method to obtain explicit expression of design variables and performance function before implement our proposed method There are three main steps to build surrogate model: 1) Design reasonable simulation experiments. 2) obtain data set of design variables and performance function by implemening simulation experiments. 3) build surrogate model to obtain explicit expression of design variables and performance function by using data sets. The common surrogate models include polynomial response surface, support vector machine and kriging model et al. In practical applications, physical problems of the disciplines may have different degrees of nonlinearity and different levels of digital noises, the selection of the surrogate model depends on specific optimization problems. In addation, the surrogate model is not only a method to obtain explicit expression, but also a method to significantly reduce the computation time and simplify the complexity of optimization model with coupled subsystems under correlated uncertainties.

5. Case Studies

In this section, both a mathematical example and a case study of an engineering system are provided to illustrate feasibility and validity of the proposed method.

5.1 A mathematical example

To illustrate the proposed MDO method, we first consider a classic MDO problem [29], which has two subsystems. Each subsystem has a coupling state variable. Nonlinear coupling relationship exists between the two subsystems. The mathematical expression is as follows:

$$Min \qquad f = x_2^2 + x_3 + y_1 + e^{-y_2}$$

where

$$y_1 = x_1^2 + x_2 + x_3 - 0.2 y_2$$
$$y_2 = \sqrt{y_1} + x_1 + x_3$$

$$s.t. \qquad g_1 = (\frac{y_1}{8}) - 1 \geq 0$$

$$g_2 = 1 - (\frac{y_2}{10}) \geq 0$$

The lower and upper bounds of design variables x_1, x_2, $x_3$and coupling state variablesy_1, $y_2$are shown in Table 1.

Table 1 Lower and upper bounds of design variables

	x_1	x_2	x_3	y_1	y_2
value range	[-10,10]	[0,10]	[0,10]	[3.5,∞)	(-∞,24]

The mean valuesofx_1,x_2,$x_3$are 0,5,5, respectively. To demonstrate and understand effects of correlated uncertainties on optimization results, we consider three example cases of the correlation coefficient matrix of the three uncertaintiesx_1, x_2, x_3:

(1) Case 1:x_1, x_2 are positively correlated; x_1, x_3 are negatively correlated; x_2, x_3 are positively correlated.

$$\rho_1 = \begin{pmatrix} 1 & 0.2 & -0.1 \\ 0.2 & 1 & 0.3 \\ -0.1 & 0.3 & 1 \end{pmatrix}$$

(2)Case 2:x_1, x_2, x_3 are positively correlatedwith each other.

$$\rho_2 = \begin{pmatrix} 1 & 0.2 & 0.1 \\ 0.2 & 1 & 0.3 \\ 0.1 & 0.3 & 1 \end{pmatrix}$$

(3) Case 3:x_1, x_2, x_3 are negatively correlatedwith each other.

$$\rho_3 = \begin{pmatrix} 1 & -0.2 & -0.1 \\ -0.2 & 1 & -0.3 \\ -0.1 & -0.3 & 1 \end{pmatrix}$$

According to Eq. (2),the covariance matricesunder the above three cases can be calculated in combination with the interval theory as:

$$C_1 = \begin{pmatrix} 100/3 & 10/3 & -5/3 \\ 10/3 & 25/3 & 5/2 \\ -5/3 & 5/2 & 25/3 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} 100/3 & 10/3 & 5/3 \\ 10/3 & 25/3 & 5/2 \\ 5/3 & 5/2 & 25/3 \end{pmatrix}$$

$$C_3 = \begin{pmatrix} 100/3 & -10/3 & -5/3 \\ -10/3 & 25/3 & -5/2 \\ -5/3 & -5/2 & 25/3 \end{pmatrix}$$

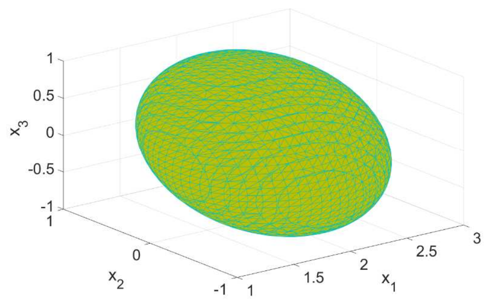
Then according to Eq. (10),we construct ellipsoidal models of x_1,x_2,$x_3$as

$$G_1 = (X-\bar{X})^T C^{-1}(X-\bar{X}) = \begin{pmatrix} x_1 & - & 0 \\ x_2 & - & 5 \\ x_3 & - & 5 \end{pmatrix}^T \begin{pmatrix} 100/3 & 10/3 & -5/3 \\ 10/3 & 25/3 & 5/2 \\ -5/3 & 5/2 & 25/3 \end{pmatrix}^{-1} \begin{pmatrix} x_1 & - & 0 \\ x_2 & - & 5 \\ x_3 & - & 5 \end{pmatrix} \le \varepsilon_1^2$$

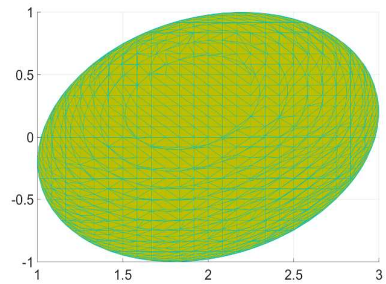
$$G_2 = (X-\bar{X})^T C^{-1}(X-\bar{X}) = \begin{pmatrix} x_1 & - & 0 \\ x_2 & - & 5 \\ x_3 & - & 5 \end{pmatrix}^T \begin{pmatrix} 100/3 & 10/3 & 5/3 \\ 10/3 & 25/3 & 5/2 \\ 5/3 & 5/2 & 25/3 \end{pmatrix}^{-1} \begin{pmatrix} x_1 & - & 0 \\ x_2 & - & 5 \\ x_3 & - & 5 \end{pmatrix} \le \varepsilon_2^2$$

$$G_3 = (X-\bar{X})^T C^{-1}(X-\bar{X}) = \begin{pmatrix} x_1 & - & 0 \\ x_2 & - & 5 \\ x_3 & - & 5 \end{pmatrix}^T \begin{pmatrix} 100/3 & -10/3 & -5/3 \\ -10/3 & 25/3 & -5/2 \\ -5/3 & -5/2 & 25/3 \end{pmatrix}^{-1} \begin{pmatrix} x_1 & - & 0 \\ x_2 & - & 5 \\ x_3 & - & 5 \end{pmatrix} \le \varepsilon_3^2$$

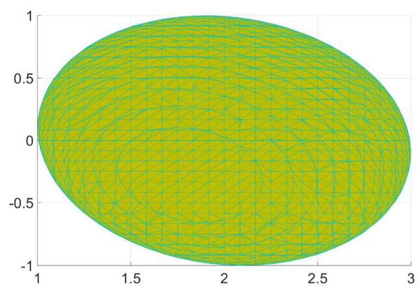
According to the minimum volume method [23, 28], we can get $\varepsilon_1=\varepsilon_2=\varepsilon_3=1.732$. As an illustrating example, correlations of the three uncertainties under case 1 are shown in Fig. 5(a), and correlations between any two of the three uncertainties are shown in Fig. 5 (b),(c),(d).



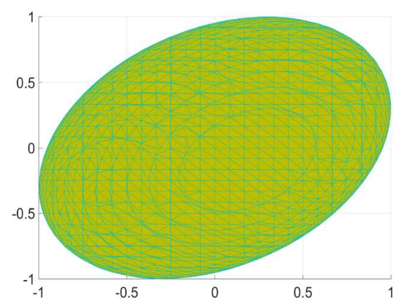
(a) x_1-x_2-x_3



(b) x_1-x_2



(c) x_1-x_3



(d) x_2-x_3

Fig. 4 Schematic diagram of correlated uncertainties

The CO model under correlated uncertainties for case 1 is as follows:

System level:

$$
\begin{aligned}
&Min: \ f = x_2^2 + x_3 + y_1 + e^{-y_2} \\
&s.t. \quad J_1 \le e, \quad J_2 \le e \\
&X_0 = [x_1, x_2, x_3, y_1, y_2]
\end{aligned}
$$

Subsystem 1:

$$
\begin{aligned}
Min:\quad & J_1=(x_{11}-x_1)^2+(x_{12}-x_2)^2+(x_{13}-x_3)^2 \\
& \qquad +(y_{11}-y_1)^2+(y_{12}-y_2)^2 \\
s.t.\quad & -10\le x_{11}\le 10,\ 0\le x_{12}\le 10 \\
& 0\le x_{13}\le 10,\ g_1, g_2, \\
& G_1\le \varepsilon_1^{\ 2} \\
X_1= & [x_{11}, x_{12}, x_{12}, y_{11}, y_{12}]
\end{aligned}
$$

Subsystem 2:

$$
\begin{aligned}
Min:\quad & J_2=(x_{21}-x_1)^2+(x_{23}-x_3)^2 \\
& \qquad +(y_{21}-y_1)^2+(y_{22}-y_2)^2 \\
s.t.\quad & -10\le x_{21}\le 10,\ 0\le x_{23}\le 10 \\
& g_1, g_2, G_1\le \varepsilon_1^{\ 2} \\
X_2= & [x_{21}, x_{23}, y_{21}, y_{22}]
\end{aligned}
$$

In subsystem 1, G_1 is a constraint to describe the correlations between x_{11},x_{12} and x_{13} under case 1. When considering correlations under case 2 and case 3, $G_2\le\varepsilon_2^{\ 2}$,$G_3\le\varepsilon_3^{\ 2}$ should be used to substitute $G_1\le\varepsilon_1^{\ 2}$,respectively. The value of compatibility constraint e is 0.001. The corresponding CO calculation framework is shown in Fig. 5.

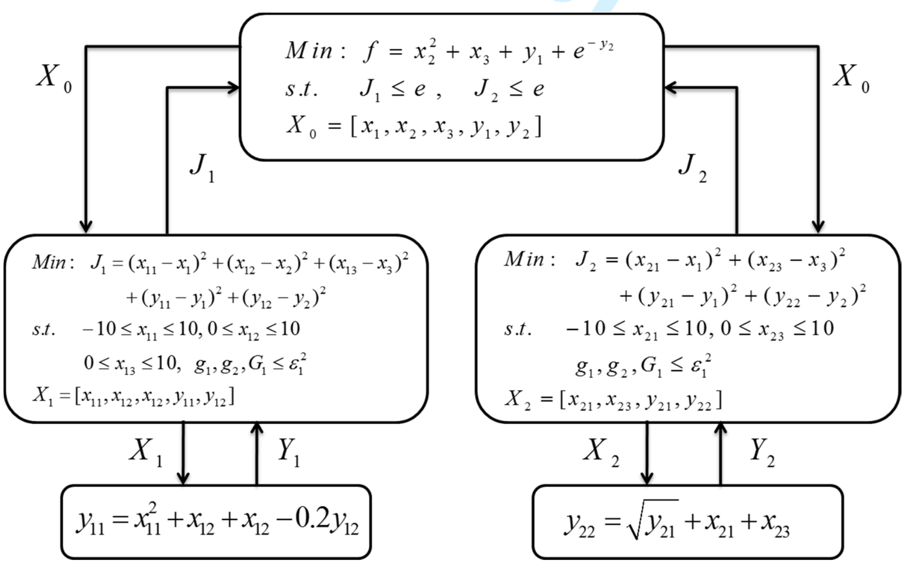


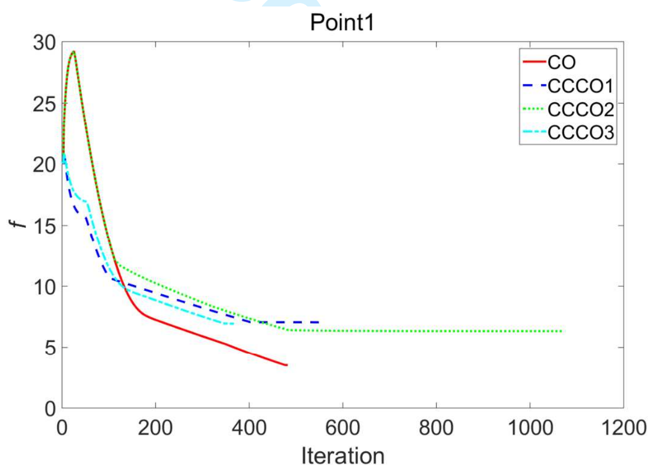
Fig. 5 CO framework of the mathematical example (case 1)

Optimization results can be sensitive to the initial point, particularly, the optimization result of the CO method tends to converge to the local optimal solution close to the initial point [30-32]. Three initial points are selected in this example (Table 2) to illustrate such effects.

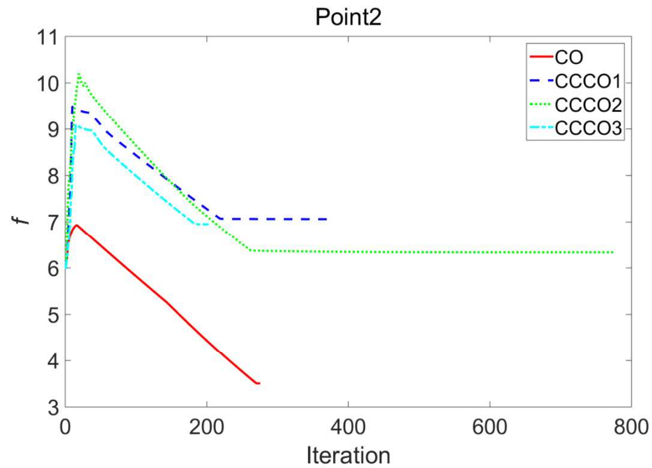
Table 2 Initial points

	x_1	x_2	x_3	y_1	y_2	f
Point 1	1	2	5	10	4	20.1059
Point 2	5	0	0	6	7	6.0009
Point 3	2	2	2	5	8	11.0003

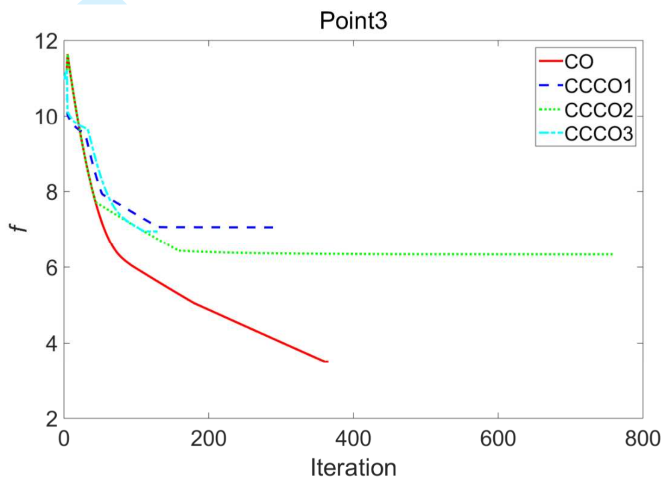
The optimization iteration procedures with initial points 1, 2, and 3are shown in Fig.6 (a), (b), and (c), respectively.



(a)



(b)



(c)

Fig. 6 Optimization iteration procedures (CO: conventional MDO method without considering correlated uncertainties; CCCO1, CCCO2, CCCO3: proposed MDO using ρ_1, ρ_2, ρ_3 as the correlation coefficient matrix of three uncertainties, respectively)

The optimization results using the three initial points are shown in Table 3.From Table 3 we can see that for different initial points they take different numbers of iterations (n columns) to converge to the same optimization result (in f column). The initial point can affect the calculation efficiency.

Table 3 Optimization results

	x_1	x_2	x_3	y_1	y_2	f	n		
							Point1	Point2	Point3
CO	2.9218	0.0003	0	3.5	4.8480	3.5078	480	274	364
CCCO1	2.7309	0.5380	3.2705	3.5	7.8175	7.0603	565	381	292
CCCO2	2.6358	0.9509	1.9294	3.5	6.3813	6.3353	1071	776	761
CCCO3	2.8763	0.0001	3.4501	3.5	8.2526	6.9504	365	201	128

Table4 summarizes the difference (in percentage) of results obtained using the proposed MDO method and those obtained using the conventional MDO in terms of the optimization variation defined as:

$$\text{Optimization variation} = \frac{f_{CCCO} - f_{CO}}{f_{CO}} \times 100\% \cdot$$

Apparently correlations between uncertainties affect the optimization results. Therefore, for accurate and informative optimization results in practical engineering, correlations must be considered during the MDO procedure; ignoring the correlations between uncertainties leads to inaccurate, actually optimistic results that may mislead the system designs.

Table 4 Optimization variations (unit: %)

	CCCO1	CCCO 2	CCCO3	mean
	101.27	80.61	98.14	93.34

From the optimization results in Tables 3, we can see that the best-obtained result of CO method is 3.5078; of CCCO1 method is 7.0603; of CCCO2 method is 6.3353; of CCCO3 method is 6.9504. Among the three example cases considered, when the three uncertainties are all positively correlated (CCCO2), the optimization result is the best; when the three uncertainties have mixed positive and negative correlations (CCCO1), the optimization result appears the worst for the considered parameter values.

However, this example only has three design variables. As for a large number of design variables, the original method needs to calculate all correlation coefficients among design variables, which is cumbersome and may not seize the essence of the optimizaition problem. In such cases, principal component analysis method may be used. Firstly, all design variables are grouped according to different disciplines; Secondly, linear combinations of design variables should be obtained for each group. Finally, the correlations between linear combinations of design variables are calculated. However, it is worth mentioning that the proposed advanced CO algorithm may be not suitable when there are a large number of variables.This is mainly because a lot of coupling variables will lead to difficulty to satisfy the compatibility constraints, which is still a challenging problem in MDO.

5.2 An engineering example

Consider a speed reducer structure design problem [33]. Figure 7 illustrates the speed reducer diagram. The design objective is to minimize the volume of the structure (objective function). The main constrains are the bending stress and contact stress of the gear tooth, the torsional deformation and stress requirements of the shaft.

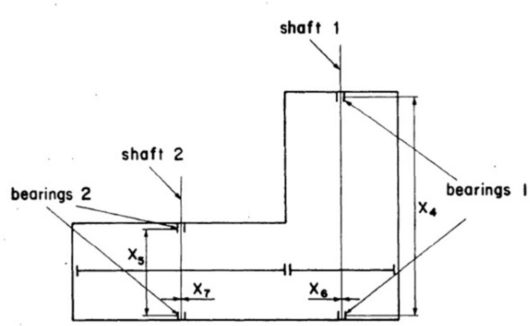


Fig. 7 The structural diagram of speed reducer

The mathematical optimization model of speed reducer is as follows:

$$\begin{aligned} Min \quad f &= 0.7854x_1x_2^2(3.3333x_3^2 + 14.9334x_3 - 43.0934) - 1.5079x_1(x_6^2 + x_7^2) \\ &\quad + 7.477(x_6^3 + x_7^3) + 0.7854(x_4x_6^2 + x_5x_7^2) \end{aligned}$$

$$
\begin{aligned}
s.t.\quad & g_1 = 27/(x_1 x_2^3 x_3) - 1 \le 0 \\
& g_2 = 397.5/(x_1 x_2^3 x_3^2) - 1 \le 0 \\
& g_3 = 1.93 x_4^2/(x_2 x_3 x_6^4) - 1 \le 0 \\
& g_4 = 1.93 x_5^2/(x_2 x_3 x_7^4) - 1 \le 0 \\
& g_5 = 10\sqrt{(745 x_4/(x_2 x_3))^2 + 1.69\times 10^7}/x_6^3 - 1100 \le 0 \\
& g_6 = 10\sqrt{(745 x_5/(x_2 x_3))^2 + 1.575\times 10^8}/x_7^3 - 850 \le 0 \\
& g_7 = (1.5 x_6 + 1.0)/x_4 - 1 \le 0 \\
& g_8 = (1.1 x_7 + 1.0)/x_5 - 1 \le 0 \\
& g_9 = x_2 x_3 - 40 \le 0 \\
& g_{10} = 5 - x_1/x_2 \le 0 \\
& g_{11} = x_1/x_2 - 12 \le 0
\end{aligned}
$$

Wherex_1 is gear tooth width, x_2 is the teeth module, x_3 is the number of teeth of the pinion,x_4 is the distance between the bearing1, x_5 is the distance between bearing 2, x_6 is the diameter of the shaft 1, x_7 is the diameter of the shaft 2, g_1, g_2 are constraints of gear tooth bending stress and contact stress, g_3~g_8 are constraints of the shaft deformation, stress et al, g_9~$g_{11}$are geometric constraints. The bounds of each variable are shown in Table5. Moreover, if the case contains implicit equations of the constraint equations, it is simple and effective to use the surrogate model technology to transform into explicit expression.

Table 5 The lower and upper bounds of design variables(unit: mm)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
value range	[2.6 3.6]	[0.7 0.8]	[17 23]	[7.3 8.3]	[7.3 8.3]	[2.9 3.9]	[5.0 5.5]

According to [34],

$$x_1 = \varphi_d x_2 x_3$$

Whereφ_d is the coefficient of tooth width,x_1, x_2, x_3 are uniform distributed with mean values of 3.1, 0.75 and 20, respectively. x_1, x_2, x_3 have certain correlations and the corresponding correlation coefficient matrix is

$$\rho = \begin{pmatrix} 1 & 0.12 & 0.20 \\ 0.12 & 1 & -0.24 \\ 0.20 & -0.24 & 1 \end{pmatrix}$$

The covariance matrix is obtained as:

$$C=\begin{pmatrix} D(X_1) & Cov(X_1,X_2) & Cov(X_1,X_3) \\ Cov(X_2,X_1) & D(X_2) & Cov(X_2,X_3) \\ Cov(X_3,X_1) & Cov(X_3,X_2) & D(X_3) \end{pmatrix}=\begin{pmatrix} 1/12 & 0.001 & 0.1 \\ 0.001 & 0.01/12 & -0.012 \\ 0.1 & -0.012 & 3 \end{pmatrix}$$

Then according to (10), the ellipsoidal model is obtained as:

$$G=(X-\bar{X})^T C^{-1}(X-\bar{X})=\begin{pmatrix} x_1 & - & 3.1 \\ x_2 & - & 0.75 \\ x_3 & - & 20 \end{pmatrix}^T \begin{pmatrix} 1/12 & 0.001 & 0.1 \\ 0.001 & 0.01/12 & -0.012 \\ 0.1 & -0.012 & 3 \end{pmatrix}^{-1} \begin{pmatrix} x_1 & - & 3.1 \\ x_2 & - & 0.75 \\ x_3 & - & 20 \end{pmatrix} \le \varepsilon^2$$

According tothe CO strategy, the optimization model of speed reducer is decomposed into three subsystems. The CO model under the condition of correlated uncertainties is as follows:

System level:

$$f=f_1+f_2+f_3$$

$$\begin{aligned} J_1 &= (x_1-x_{11})^2+(x_2-x_{12})^2+(x_3-x_{13})^2 \\ J_2 &= (x_1-x_{21})^2+(x_4-x_{24})^2+(x_6-x_{26})^2 \\ J_3 &= (x_1-x_{31})^2+(x_5-x_{35})^2+(x_7-x_{37})^2 \end{aligned}$$

Subsystem 1:

$$f_1=0.7854x_{11}x_{12}^2(3.3333x_{13}^2+14.9334x_{13}-43.0934)$$

$$J_1=(x_1-x_{11})^2+(x_2-x_{12})^2+(x_3-x_{13})^2$$

$$\begin{aligned} s.t.\quad & g_1=27/(x_{11}x_{12}^3x_{13})-1\le 0 \\ & g_2=397.5/(x_{11}x_{12}^3x_{13}^2)-1\le 0 \\ & g_9=x_{12}x_{13}-40\le 0 \\ & g_{10}=5-x_{11}/x_{12}\le 0 \\ & g_{11}=x_{11}/x_{12}-12\le 0 \\ & G=(X-\bar{X})^T C^{-1}(X-\bar{X})\le\varepsilon^2 \end{aligned}$$

Subsystem 2:

$$f_2=7.477x_{26}^3+0.7854x_{24}x_{26}^2-1.5079x_{21}x_{26}^2$$

$$J_2 = (x_1 - x_{21})^2 + (x_4 - x_{24})^2 + (x_6 - x_{26})^2$$

$$
\begin{aligned}
s.t.\quad & g_1 = 27/(x_{21}x_{22}^3 x_{23}) - 1 \le 0 \\
& g_2 = 397.5/(x_{21}x_{22}^3 x_{23}^2) - 1 \le 0 \\
& g_3 = 1.93x_4^2/(x_{22}x_{23}x_{26}^4) - 1 \le 0 \\
& g_5 = 10\sqrt{(745x_{24}/(x_{22}x_{23})^2 + 1.69\times 10^7}/x_{26}^3 - 1100 \le 0 \\
& g_7 = (1.5x_{26} + 1.0)/x_{24} - 1 \le 0 \\
& g_9 = x_{22}x_{23} - 40 \le 0 \\
& g_{10} = 5 - x_{21}/x_{22} \le 0 \\
& g_{11} = x_{21}/x_{22} - 12 \le 0 \\
& G = (X - \bar{X})^T C^{-1} (X - \bar{X}) \le \varepsilon^2
\end{aligned}
$$

Subsystem 3:

$$f_3 = 7.477x_{37}^3 + 0.7854x_{35}x_{37}^2 - 1.5079x_{31}x_{37}^2$$

$$J_3 = (x_1 - x_{31})^2 + (x_5 - x_{35})^2 + (x_7 - x_{37})^2$$

$$
\begin{aligned}
s.t.\quad & g_1 = 27/(x_{31}x_{32}^3 x_{33}) - 1 \le 0 \\
& g_2 = 397.5/(x_{31}x_{32}^3 x_{33}^2) - 1 \le 0 \\
& g_4 = 1.93x_{35}^2/(x_{32}x_{33}x_{37}^4) - 1 \le 0 \\
& g_6 = 10\sqrt{(745x_{35}/(x_{32}x_{33})^2 + 1.575\times 10^8}/x_{37}^3 - 850 \le 0 \\
& g_8 = (1.1x_{37} + 1.0)/x_{35} - 1 \le 0 \\
& g_9 = x_{32}x_{33} - 40 \le 0 \\
& g_{10} = 5 - x_{31}/x_{32} \le 0 \\
& g_{11} = x_{31}/x_{32} - 12 \le 0 \\
& G = (X - \bar{X})^T C^{-1} (X - \bar{X}) \le \varepsilon^2
\end{aligned}
$$

The corresponding CO calculation framework is shown in Fig. 8, where the value of compatibility constraint is e=0.01,the ellipsoidal minimum radius ε is 2.191.

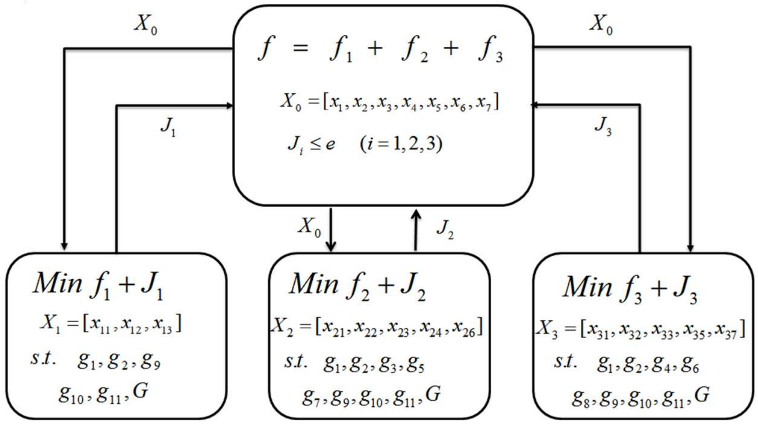


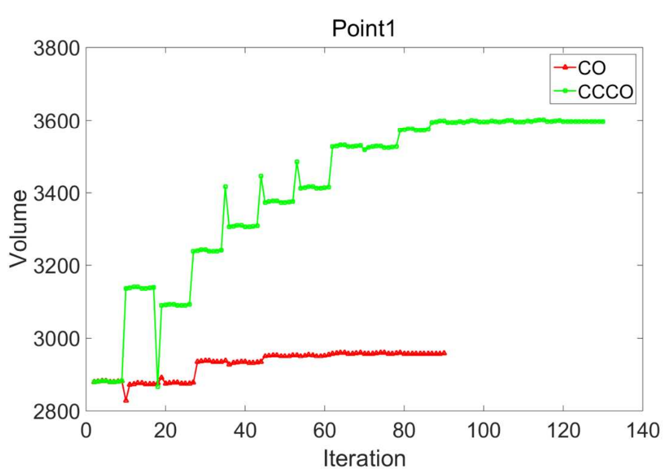
Fig. 8 CO framework of speed reducer

Three initial design points are shown in Table6.

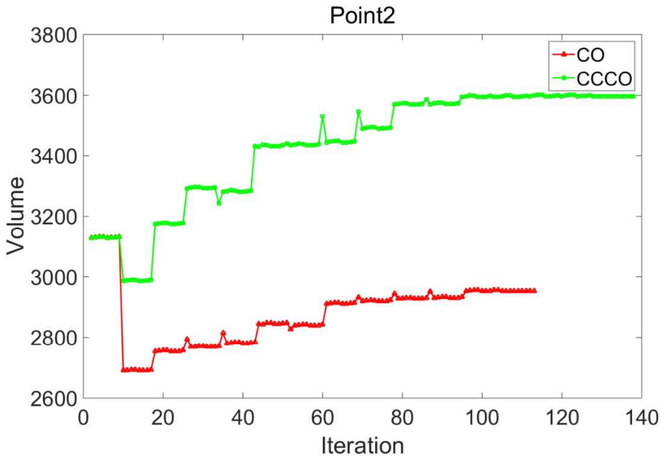
Table 6 Initial design points

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	f
Point1	3.5	0.70	17	7.3	7.715	3.35	5.099	2879.1
Point2	3.0	0.75	18	7.5	7.500	3.40	5.200	3128.4
Point3	3.2	0.78	20	7.9	7.599	3.20	5.100	3709.9

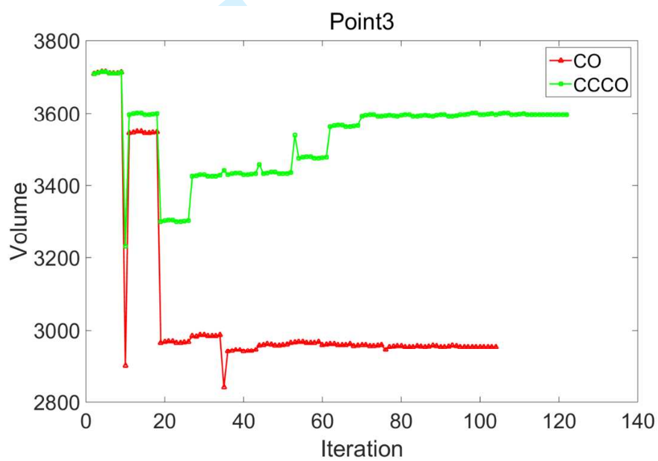
The optimization iteration procedures of conventional MDO and proposed MDOare shown in Fig. 9, and the corresponding optimization results areshown in Table7.



(a)



(b)



(c)

Fig.9 Optimization iteration procedures of speed reducer (CO: conventional MDO method without considering correlated uncertainties; CCCO: proposed MDO using ρ as the correlation coefficient matrix of uncertainties)

Table 7 Optimization results

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	f
Point1	CO	3.570	0.7	17	7.30	7.713	3.276	5.216	2959.1
	CCCO	3.564	0.7	20	7.30	7.711	3.279	5.218	3527.5
	CO	3.568	0.7	17	7.30	7.714	3.271	5.210	2953.3

Point2	CCCO	3.564	0.7	20	7.30	7.709	3.279	5.218	3528.0
Point3	CO	3.569	0.7	17	7.30	7.713	3.275	5.211	2955.3
	CCCO	3.564	0.7	20	7.30	7.708	3.279	5.217	3527.4

From Table 7 we can see that the proposed method using different initial points generates optimization results with smaller variations than the conventional CO method for this example system. In addition, compared with optimization results obtained by the conventional CO method, values of the objective function increase 19.21%, 19.46%, and 19.36%, respectively under the three initial points by using our proposed method. This is consistent with the observation made in the mathematical example that the conventional MDO method tends to give inaccurate and optimistic results that can mislead the system design activities, while our proposed method generates results that are more informative and applicable to the engineering reality.

6. Conclusions and Future Work

In this paper characteristics of correlated uncertainties are investigated, and a new quantitative model of correlated uncertainties is established using the ellipsoidal model and the interval theory. Based on the constructed uncertainty correlation model, a new MDO method is proposed to consider effects of correlated uncertainties on the system optimization. Both a mathematical example and an engineering example are analyzed to illustrate the feasibility and validity of the proposed method. The proposed method belongs to non-probabilistic MDO methods and is suitable to solve design problems with a given range of uncertainties. In this paper, uncertainties are considered to be independent of each other. If the uncertainties are coupled (not independent of each other), it may fail to establish the correlated model of uncertainties. However, it would be a feasible method to deal with the coupled uncertainties by combining system sensitivity analysis with Rosenblatt transformation, Nataf transformation or orthogonal transformation. Hence, how to model coupled uncertainties is the key research in the future work. Moreover, we only consider correlations of static uncertainties in this paper. In our future work we will investigate how to quantify correlations of dynamic uncertainties in mechanical systems and new models to consider their effects in the MDO procedures.

Acknowledgments

This research is supported by the National Natural Science Foundation of China under the contract number 51475082and 11302044.

References

[1]Timothy, W.S., Joaquim, R.R.A.M., 2011,"Multidisciplinary design optimization for complex engineered systems: report from a national science foundation workshop [J]", ASME J. Mech. Des., 133(10), 101002.

[2]Mathieu, B., Nicolas B., Philippe, D., Abdelhamid, C., 2012, "A survey of multidisciplinary design optimization methods in launch vehicle design [J]",Struct. Multidiscip. Optim.,45(5), pp. 619-642.

[3]Sobieszczanski-Sobieski, J., Haftka, R.T., 1997,"Multidisciplinary aerospace design optimization: survey of recent developments [J]", Structural Optimization 14(1),pp. 1-23.

[4]Joaquim, R.R.A.M., Andrew, B.L., 2013,"Multidisciplinary design optimization: a survey of architectures [J]", AIAA journal, 51(9),pp.2049-2075.

[5]Safavi, E., Tarkian, M., Gavel, H., et al., 2015,"Collaborative multidisciplinary design optimization: A framework applied on aircraft conceptual system design [J]", Concurrent Engineering 23(3),pp.236-249.

[6]Meng, D.B., Li, Y.F., Huang H.Z.,,et al., 2014,"Reliability-based multidisciplinary design optimization using subset simulation analysis and its application in the hydraulic transmission mechanism design [J]", Journal of Mechanical Design, 137(5),pp.051402-1-9.

[7]Du, X., and Chen, W., 2000,"Methodology for managing the effect of uncertainty in simulation-based design [J]", AIAA Journal, 38(8),pp. 1471-1478.

[8]Du, X, and Chen, W., 2002,"Efficient uncertainty analysis methods for multidisciplinary robust design [J]", AIAA Journal, 40(3),pp. 545-552.

[9]Koch, P. N., Simpson, T.W., Allen, J.K., et al., 1999, "Statistical approximations for multidisciplinary design optimization: the problem of size [J]", Journal of Aircraft, 36(1),pp. 275-286.

[10]Gu, X., Renaud, J.E., Batill, S.M., Brach, R.M., Budhiraja, A.S.,2000, "Worst case propagated uncertainty of multidisciplinary systems in robust design optimization [J]",Struct. Multidiscip. Optim., 20(3),pp.190-213.

[11]Gu, X., Renaud, J.E., 2002, "Implementation study of implicit uncertainty propagation (IUP) in decomposition-based optimization [J]", AIAA-2002-5416, Proceedings of the 9th AIAA/USAF/NASA/ISSMO Symposium and Exihibit on Multidisciplinary Analysis and Optimization, Atlanta, GA,Sep. 4-6, pp. 32-39

[12]Harish, A., John, E. R., Evan, L. P., Dhanesh, P., 2004, "Uncertainty quantification using evidence theory in multidisciplinary design optimization [J]", Reliability Engineering and System Safety, 85,pp.281-294.

[13]Chu X.Z., Gao L., Li W.D., 2009,"An uncertainty analysis approach to multidisciplinary design optimization [J]", Concurrent Engineering 17, pp.121-128.

[14]Noh, Y., Choi, K. K., Du L., 2009, "Reliability-based design optimization of problems with correlated input variables using a Gaussian Copula [J]",Struct. Multidiscip. Optim., 38(1), pp. 1-16.

[15]Song, S., Lu, Z., Song Z, 2011, "Reliability sensitivity analysis involving correlated random variables by directional sampling [C]" ,Proceedings of 2011 International Conference on Quality, Reliability, Risk, Maintenance, and Safety Engineering, Xi'an, China

[16]Song S., Lu Z., 2010, "Directional simulation based reliability sensitivity analysis for multiple design points / failure modes system with normal correlative variables [J]",ActaAeronautica et AstronauticaSinica, 31, pp. 109-118.

[17]Vicino, A.,Tesi, A., Milanese, M., 1990, "Computation of nonconservative stability perturbation bounds for systems with nonlinearly correlated uncertainties [J]",IEEE transaction on automatic control, 35(7),pp.835-841.

[18]Ben-Haim, Y., Elishakoff, I., 1990,"Convex Models of uncertainty in applied mechanics [M]", Elsevier Science,Amsterdam.

[19]Ben-Haim Y. 1994,"A non-probabilistic concept of reliability [J]" , Structure and safety,14,pp.227-245.

[20]Ben-Haim, Y., 1996, "Robust reliability in the mechanical sciences [M]", Springer-Verlag Berlin Heidelberg NewYork.

[21]Li, Y., Huang, H.,Liu, Z., 2004,"Convex model in robust reliability analysis of structure [J]", Journal of Basic Science and Engineering,12(4),pp.383-390.

[22]Jiang, C., Han, X., Lu, G.Y., et al., 2011,"Correlation analysis of non-probabilistic convex model and corresponding structural reliability technique [J]", Computer Methods Appl. Mech. Engrg. 200,pp.2528-2546.

[23]Elishakoff, I.,1999, "Whys and hows in uncertainty modeling: probability, Fuzziness and Anti-optimization [M]", Springer, Vienna.

[24]Ball, K.,1997,"An elementary introduction to modern convex geometry [M]", In Favors of Geometry,Volume MSRI Publ.31.Cambrige Univ.Press.

[25]Kroo, I., Altus, S., Braun, R., Gage, P., and Sobieski, I., 1994, "Multidisciplinary Optimization Methods for Aircraft Preliminary Design [C]", AIAA 94-4325, Fifth AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Analysis and Optimization, September 7-9, Panama City, Florida.

[26]Kroo, I., 1995,"Decomposition and Collaborative Optimization for Large Scale Aerospace Design [C]", ICASE/LaRC/SIAM Workshop on Multidisciplinary Optimization.

[27]Meng D.B., Zhang X.L., Yang Y.J., et al., 2016,"Interaction balance optimization in multidisciplinary design optimizations problems [J]", Concurrent Engineering 24(1),pp.48-57.

[28]Zaguskin, V.L., 1958,"On circumscribed and inscribed ellipsoids of the external volume [J]", Uspekhi Mat,Nauk,13(6),pp.89-93.

[29]Chan, L., 2001,"Evaluation of two concurrent design approaches in multidisciplinary designoptimization [C]", NRC report LM-A-077.

[30]Lin, J. G., 2004,"Analysis and enhancement of collaborative optimization for multidisciplinary design [J]", AIAA Journal,42(2),pp.348-360.

[31]Alexandrov, N.M., Lewis, R.M., 2002,"Analytical computational aspects of collaborative optimization for multidisciplinary design [J]",AIAA Journal,40(2),pp.301-309.

[32]Li, H.Y, 2010,"Research on uncertainty and multi-objectivity of multidisciplinary collaborative optimization [D]" ,Doctoral dissertation, Northeastern University, Shenyang.

[33]Shayanfar, M., Abbasnia, R., Khodam, A., 2014, "Development of a GA-based method for reliability-based optimization of structures with discrete and continuous design variables using OpenSees and Tcl [J]", Finite Elements in Analysis and Design, 90, pp. 61-73.

[34] Liang ,G.P., Ming, G.J., 2006, "Design of Machinery [M]", Higher Education Press.